Inequality involving equilateral triangles

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Let ABC be an equilateral triangle with side-length a and let P be any point inside the triangle. Prove that

$$a^2/2 \geq xPA + yPB + zPC \geq 2(xy + yz + zx)$$

where x, y, z denote the distances from P to the sides BC, CA, AB, respectively.

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Since the letters x, y, z will be needed for other purposes we will use common notation d_a, d_b, d_c for the distances from P to the sides BC, CA, AB, respectively.

Then inequality of the problem in such notation becomes

(**1**) $a^2/2 \geq \sum d_a \cdot PA \geq 2\sum d_a d_b$.

Let F_a, F_b, F_c, F be areas of $\triangle PBC, \triangle PCA, \triangle PBC, \triangle ABC,$ respectively and let $x := \dfrac{F_a}{F}$, $y := \dfrac{F_b}{F}, z := \dfrac{F_c}{F}$ be baricentric coordinates of P (with respect to $\triangle ABC$), that is $x, y, z \geq 0$ and $x + y + z = 1.$ Since $PA = (y+z)AK,$ where AK is a cevian from vertex A which contains point P and $\dfrac{BK}{KC} = \dfrac{F_c}{F_b} = \dfrac{z}{y}$ then by Stewart's formula

$AK^2 = \dfrac{y}{y+z}b^2 + \dfrac{z}{y+z}c^2 - \dfrac{yz}{(y+z)^2}a^2 = \dfrac{1}{(y+z)^2}(y(y+z)b^2 + z(y+z)c^2 - yza^2) =$

$\dfrac{1}{(y+z)^2}(y^2b^2 + z^2c^2 + yz(b^2 + c^2 - a^2))$ then $PA = \sqrt{y^2b^2 + z^2c^2 + yz(b^2 + c^2 - a^2)}$.

Also note that $d_a = \dfrac{2F_a}{a} = \dfrac{2xF}{a}$. Hence,

$$\sum d_a \cdot PA = \frac{2F}{a}\sum x\sqrt{y^2b^2 + z^2c^2 + yz(b^2 + c^2 - a^2)}$$

and since in our case $a = b = c, F = \dfrac{a^2\sqrt{3}}{4}$ then $\sqrt{y^2b^2 + z^2c^2 + yz(b^2 + c^2 - a^2)} =$ $a\sqrt{y^2 + yz + z^2}$ and $\sum d_a \cdot PA = \dfrac{a^2\sqrt{3}}{2}\sum x\sqrt{y^2 + yz + z^2}$.

Thus, LHS of (**1**) becomes $\dfrac{a^2}{2} \geq \dfrac{a^2\sqrt{3}}{2}\sum x\sqrt{y^2 + yz + z^2} \Leftrightarrow \sum x\sqrt{y^2 + yz + z^2} \leq \dfrac{1}{\sqrt{3}}$.

By Cauchy Inequality $\sum x\sqrt{y^2 + yz + z^2} = \sum \sqrt{x}\sqrt{xy^2 + xyz + xz^2} \leq$

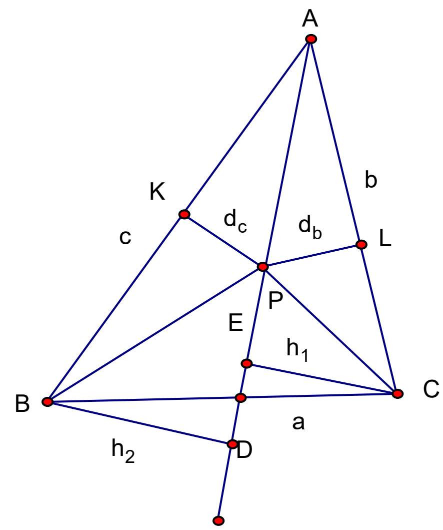
$\sqrt{x+y+z} \cdot \sqrt{\sum(xy^2 + xyz + xz^2)} = \sqrt{(x+y+z)(xy+yz+zx)} = \sqrt{xy+yz+zx}$

and since $xy + yz + zx \leq \dfrac{(x+y+z)^2}{3} = \dfrac{1}{3}$ we obtain $\sum x\sqrt{y^2 + yz + z^2} \leq \dfrac{1}{\sqrt{3}}$.

RHS of (**1**), unlike LHS of (**1**), holds for any triangle.

Indeed, since* $PA \geq \dfrac{bd_b + cd_c}{a}$ then $\sum d_a \cdot PA \geq \sum d_a \cdot \dfrac{bd_b + cd_c}{a} = \sum \dfrac{bd_ad_b + cd_cd_a}{a} =$ $\sum d_a d_b\left(\dfrac{b}{a} + \dfrac{a}{b}\right) \geq \sum 2d_a d_b = 2\sum d_a d_b$.

* **Proof** of inequality $PA \geq \dfrac{bd_b + cd_c}{a}$



From similarity $\triangle PLA \sim \triangle CDA$ and $\triangle PKA \sim \triangle BEA$ we obtain,respectively:
$\frac{d_b}{R_a} = \frac{h_1}{b} \Leftrightarrow h_1 R_a = bd_b$ and $\frac{d_c}{R_a} = \frac{h_2}{c} \Leftrightarrow h_2 R_a = cd_c$.
Hence, $R_a(h_1 + h_2) = bd_b + cd_c$ and since $a \geq h_1 + h_2$ we finally obtain
$aR_a \geq R_a(h_1 + h_2) = bd_b + cd_c$.
Equality holds iff $AP \perp BC$.